ECON 582: Chapter 16. Self-insurance

Instructor: Dmytro Hryshko

We'll cover Sections 16.3 and 16.4.

"Self-insurance" occurs when the agent uses savings/borrowing to insure himself against income fluctuations. We'll assume that $u(c_t)$ is strictly increasing, strictly concave, twice continuously differentiable.

We want to study optimal consumption decisions of an agent with an uneven income stream and a borrowing constraint.

Two kinds of borrowing constraints: "natural borrowing constraint" (less stringent) and an "ad-hoc," no-borrowing constraint (more stringent).

Let b_t be the amount of debt the consumer owes at time t, y_t the income endowment at time t, and a_t is the consumer's asset position at time t. Thus,

$$a_t = -b_t + y_t.$$

$b_t < 0$ if consumer is a saver. Assume that $b_0 = 0$ and $\underline{\beta(1+r) = 1}$. Let the budget constraint be

$$\begin{aligned} a_{t+1} &= (1+r)(a_t - c_t) + y_{t+1} \\ \beta a_{t+1} &= a_t - c_t + \beta y_{t+1} \\ -\beta b_{t+1} + \beta y_{t+1} &= -b_t + y_t - c_t + \beta y_{t+1} \\ c_t + b_t &= \beta b_{t+1} + y_t. \end{aligned}$$

At inequality, the last equation becomes

$$c_t + b_t \leq \beta b_{t+1} + y_t \tag{16.3.1}$$

"Natural borrowing limit"

If consumer is precluded from borrowing $b_{t+1} \leq 0$ for all t. Then $a_{t+1} = -b_{t+1} + y_{t+1} \geq y_{t+1}$, or $a_{t+1} \geq y_{t+1}$—assets at each t should be more or equal to the income endowment y_t.

What is the "natural" maximum amount of debt?
Since $c_t \geq 0$, it follows from (16.3.1) that

$$0 \leq c_t \leq \beta b_{t+1} - b_t + y_t$$

$$b_t - \beta b_{t+1} \leq y_t$$

$$b_t(1 - \beta L^{-1}) \leq y_t$$

$$b_t \leq (1 + \beta L^{-1} + \beta^2 L^{-2} + \dots)y_t$$

$$b_t \leq \sum_{j=0}^{\infty} \beta^j y_{t+j} \equiv \overline{b}_t \qquad (16.3.3)$$

Intuitively, we can borrow up to the PDV of our income/endowment stream since this amount will exactly replicate our income stream if we put it back into the "bank."

See handout next.

Note that (16.3.1) implies that

$$\sum_{j=0}^{\infty} \beta^j c_{t+j} \leq b_t + \sum_{j=0}^{\infty} \beta^j y_{t+j} \text{ for all } t.$$

Set t to 0, to obtain

$$\sum_{t=0}^{\infty} \beta^t c_t \leq b_0 + \sum_{t=0}^{\infty} \beta^t y_t \equiv \sum_{t=0}^{\infty} \beta^t y_t, \tag{16.3.4}$$

since $b_0 = 0$ by assumption.

Define $c_t = \bar{c}$ for all $t \geq 0$ such that it satisfies (16.3.4) at equality. Then

$$\sum_{t=0}^{\infty} \beta^t \bar{c} = \sum_{t=0}^{\infty} \beta^t y_t$$

$$\frac{\bar{c}}{1-\beta} = \sum_{t=0}^{\infty} \beta^t y_t. \qquad (16.3.6)$$

Under $c_t = \bar{c}$, for each t, (16.3.1) at equality becomes

$$\bar{c} + b_t = \beta b_{t+1} + y_t$$

$$y_t - \bar{c} = b_t(1 - \beta L^{-1})$$

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} - \bar{c}\left[1 + \beta L^{-1} + \beta^2 L^{-2} + \ldots\right] = b_t$$

$$\sum_{j=0}^{\infty} \beta^j y_{t+j} - \frac{\bar{c}}{1-\beta} = b_t \leq \bar{b}_t \equiv \sum_{j=0}^{\infty} \beta^j y_{t+j}.$$

Under the natural borrowing constraint, we have constant consumption for $t \geq 0$, i.e., perfect consumption smoothing.

What would happen if the consumer was not allowed to borrow at all? Intuitively, this will inhibit consumption smoothing for consumers whose incomes are steadily growing, and therefore who are naturally borrowers.

Under the no-borrowing constraint, consumption is constant between the periods t and $t+1$ if the borrowing constraint is not binding; otherwise it is growing. That is, the constraint binds when the consumer wants to shift resources to the present but is unable to do so. This happens when consumer's endowment is growing.

PROPOSITION 1. Given a borrowing constraint and a non-stochastic endowment stream, the limit of the nondecreasing consumption path is

$$\bar{c} \equiv \lim_{t\to\infty} c_t^* = \sup_t x_t \equiv \bar{c},$$

where $x_t \equiv \frac{r}{1+r} \sum_{j=0}^{\infty} (1+r)^{-j} y_{t+j}$—the annuity value of the tail of the income process starting from period t.

Example 16.3.2. Periodic endowment process

Time starts at $t = 0$ and runs to ∞. Endowment: 1 in even periods, 0—in odd periods.

$$x_t\big|_{t \text{ even}} = \frac{r}{1+r}\left[1 + 0 + \frac{1}{(1+r)^2} + 0 + \frac{1}{(1+r)^4} + \dots\right]$$
$$= \frac{r}{1+r}\left[\frac{1}{1 - \frac{1}{(1+r)^2}}\right]$$
$$= \frac{1-\beta}{1-\beta^2} = \frac{1}{1+\beta}.$$

$$x_t\big|_{t \text{ odd}} = \frac{r}{1+r}\left[0 + \frac{1}{(1+r)} + 0 + \frac{1}{(1+r)^3} + \dots\right]$$
$$= \frac{r}{1+r}\frac{1}{1+r}\left[\frac{1}{1 - \frac{1}{(1+r)^2}}\right]$$
$$= \frac{\beta(1-\beta)}{1-\beta^2} = \frac{\beta}{1+\beta}.$$

The limit of the optimal consumption is $\overline{c} = \sup_t x_t = \frac{1}{1+\beta}$. At time 0, savings will be $y_0 - c_0 = 1 - \frac{1}{1+\beta} = \frac{\beta}{1+\beta}$;
$a_1 = (1+r)s_0 + y_1 = (1+r)\frac{\beta}{1+\beta} + 0 = \frac{1}{1+\beta}$.

At time 1, consumer eats all the assets, so that
$a_2 = (1+r)(a_1 - c_1) + y_2 = (1+r)0 + 1 = 1$.

Thus, assets fluctuate between $(1+\beta)^{-1}$ and 1.

Section 17.4

Other cases. $\underline{\beta(1+r) < 1}$.
Recall that for the no-borrowing constraint, the Euler equation is:

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1}), = \text{if } a_{t+1} \geq -\phi_{t+1}.$$

If the constraint is not binding, that is, $a_{t+1} > -\phi_{t+1}$,

$$u'(c_{t+1}) = \frac{1}{\beta(1+r)}u'(c_t) > u'(c_t).$$

By strict concavity, $c_{t+1} < c_t$—consumption is declining when the household is not borrowing constrained. $\{c_t\}$ is a monotone decreasing sequence. If bounded below, say, because $c_t \geq 0$, c_t will converge as $t \to \infty$ to some limit and stay there.

If income is constant and equal to w, then $a_{t+1} = (1+r)(a_t - \overline{c}) + w$. In the limit, when the constraint is binding, $a_{t+1} = a_t = -\phi$, and $\overline{c} = w - \frac{r}{1+r}\phi$. That is, the rule is to consume income after the interest is paid on the debt at the borrowing limit.
Note that

$$\overline{c} = w - \frac{r}{1+r}\phi \geq 0,$$

and so

$$\phi \leq \frac{1+r}{r}w.$$

This is the natural borrowing limit for this specific example. If $\phi < \frac{1+r}{r}w$ we say that there is an ad-hoc borrowing limit.

When $\beta(1+r) = 1$, the Euler equation at equality is $c_t = c_{t+1}$. The budget constraint is $a_{t+1} = (1+r)(a_t - \bar{c}) + w$. This implies that $a_{t+1} = a_t = a_0$. Thus, any value of a_0 is a stationary value of a: it is optimal to roll over the initial asset level.

Summarizing, the steady state demand for assets is

$$\bar{a} = -\phi \text{ if } \beta(1+r) < 1$$

$$\bar{a} = a_0 \text{ if } \beta(1+r) = 1.$$