ECON 582: STOCHASTIC DYNAMIC PROGRAMMING (CHAPTER 16, ACEMOGLU)

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Let us introduce the stochastic (random) variable $z(t) \in Z \equiv \{z_1, \ldots, z_N\}$, with $z_1 < z_2 < \ldots < z_N$. The set Z is finite and thus compact. Let the instantaneous payoff at t be

$$U(x(t), x(t+1), z(t)),$$

where $x(t) \in X \subset R^K$, $K \geq 1$, and $U : X \times X \times Z \to R$.

Payoffs directly depend on the stochastic variable $z(t)$;

at time t, state is $x(t)$, control is $x(t+1)$;

initial values $x(0)$ and $z(0)$ are given.

Constraint: $x(t+1) \in G(x(t), z(t))$.

Suppose that $z(t)$ is a (first-order) Markov chain—current value of $z(t)$ depends only on its last period value, $z(t-1)$. That is,

$$\Pr\left[z(t)=z_j|z(0),\dots,z(t-1)\right]\equiv\Pr\left[z(t)=z_j|z(t-1)\right].$$

Example. Independently distributed r.v. is a Markov chain: $\Pr\left[z(t)=z_j|z(0),\dots,z(t-1)\right]=\Pr[z(t)=z_j]$.

Markov chains enable us to model environments where stochastic variables are correlated over time.

Markov chain

$$\Pr\left[z(t) = z_j | z(t-1) = z_i\right] = \pi_{ij},$$

for any $i, j = 1, \ldots, N$, where $\pi_{ij} \geq 0$ and

$$\sum_{j=1}^{N} \pi_{ij} = 1, \text{ for each } i = 1, \ldots, N.$$

π_{ij} is called a transition probability—the probability of the stochastic state z transitioning from state z_i to state z_j.

Optimal growth problem

The objective is to maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c(t)).$$

$c(t)$ is consumption per capita at time t; $u(\cdot)$ is the instantaneous utility function.

E_0 is the expectation conditional on information available at the beginning of time $t = 0$. Future values of c are stochastic as they depend on the realizations of future z's.

Let the production function in per capita terms be:

$$y(t) = f(k(t), z(t)),$$

where $k(t)$ is capital per worker and $z(t) \in Z \equiv \{z_1, z_2, \ldots, z_N\}$. E.g., $z(t)$ is a stochastic TFP.

The resource constraint is:

$$k(t+1) = f(k(t), z(t)) + (1-\delta)k(t) - c(t),$$

$k(0) > 0$ is given and $k(t) \geq 0$ for all t; δ is the depreciation rate. Thus, $c(t)$ will depend on $z(t)$, and, in fact, on the entire history of $z(t)$ up to t.

Define

$$z^t \equiv (z(1), \ldots, z(t)).$$

as the history of $z(t)$ up to time t. z^t does not include $z(0)$ since it's given, and includes t elements. Let $Z^t = \underbrace{Z \times Z \times \cdots \times Z}_{\text{t times}}$ so that $z^t \in Z^t$. For given $k(0)$, the level of consumption at time t can be written as

$$c(t) = c[z^t].$$

That is, consumption at time t is a function of the entire sequence of random variables observed up to t; **not** a function of future z's; those are not realized yet and we assume that we don't have advance information about them.

The resource constraint can be written as:

$$k(t+1) = f(k(t), z(t)) + (1-\delta)k(t) - c[z^t] \equiv k[z^t].$$

Using this definition we can rewrite the budget constraint as

$$k[z^t] = f(k[z^{t-1}], z(t)) + (1-\delta)k[z^{t-1}] - c[z^t],$$

for all $z^{t-1} \in Z^{t-1}$ and $z^t \in Z^t$.

We can express the maximization problem as

$$\max_{\{k[z^t]\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t U\left(k[z^{t-1}], k[z^t], z(t)\right),$$

where

$$U\left[k[z^{t-1}], k[z^t], z(t)\right] = u\left[f\left(k(t), z(t)\right) + (1-\delta)k(t) - k(t+1)\right].$$

Initial conditions: $k(0)$ and $z(0)$ given.

Expectation is over the entire set of z's.

Recursive formulation

Since $z(t)$ is a Markov chain, its realization affects not only the resources available for $k(t+1)$ and $c(t)$—via $f(k(t), z(t))$—but also contains information about distribution of $z(t+1)$.

Naturally, $k(t+1) = \pi\left(k(t), z(t)\right)$.

Recursive formulation will be

$$\begin{aligned} V(k,z) = \max_{k' \in [0, f(k,z)+(1-\delta)k]} &\{u\left[f(k,z) + (1-\delta)k - k'\right] \\ &+ \beta E[V(k', z')|z]\}, \end{aligned}$$

where $E[\cdot|z]$ is the expectation *conditional* on the current value of z and we used the fact that $z(t)$ is a Markov chain. Expectation is over the next-period z, z'.

Let's suppose that the program has a solution, that is, there exists a feasible plan that achieves the maximum value $V(k, z)$ starting with k and z. Then, $k' = \pi(k, z)$ and for any k and z,

$$V(k, z) = u\left[f(k, z) + (1-\delta)k - \pi(k, z)\right] + \beta E\left[V(\pi(k, z), z')|z\right].$$

Under certain assumptions, $\pi(k, z)$ is uniquely defined, and

$$k(t+1) = \pi(k(t), z(t)).$$

General formulation. Sequence problem

Problem B1 (sequence problem):

$$V^*(x(0), z(0)) = \max_{\{x[z^t]\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U\left(x[z^{t-1}], x[z^t], z(t)\right)$$

s.t.

$$x[z^t] \in G\left(x[z^{t-1}], z(t)\right), \text{ for all } t \geq 0$$

$x(0), z(0)$ given.

Expectation is conditioned on $z(0)$ and over all possible infinite sequences $\{z(1), z(2), \ldots\}$.

General formulation. Recursive problem

Problem B2 (recursive problem):

$$V(x,z) = \max_{x' \in G(x,z)} \{U(x,x',z) + \beta E\left[V(x',z')|z\right]\}, \text{ for all } x \in X, z \in Z.$$

$V : X \times Z \to R$ is a real-valued function, and $x' \in G(x,z)$ is the constraint that specifies feasible values of the next period x, x', given the current values of x and z.

Under certain assumptions, **Problem B1** and **Problem B2** are equivalent.

Application: The Permanent Income Hypothesis

Consumption smoothing problem of a household facing an uncertain income stream. Household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c(t)),$$

$c(t) \geq 0$ is consumption at time t. Assume that $u(\cdot)$ is strictly increasing, continuously differentiable and concave.

The household can borrow and lend freely at a constant interest rate $r > 0$, and the lifetime budget constraint is

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} c(t) \leq a(0) + \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} w(t), \tag{1}$$

where $w(t)$ is household labor income at t, and $a(0)$ is the level of initial financial assets.

Suppose that $w(t)$ is random and takes values from the set $\{w_1, \ldots, w_N\}$. Assume that $w(t)$ is independent over time and w_j is drawn with π_j so that $\sum_{i=1}^{N} \pi_i = 1$. Constraint is stochastic and must hold with probability 1 for all sample paths of $w(t)$.

Although there is no explicit constraint, there will be *endogenous borrowing constraints*. If $w_1 = 0$ with $\pi_1 > 0$, there's a positive probability that $w(t) = 0$ for all t. Thus,

$$a(t) \geq -\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} w_1 \equiv -b_1,$$

where w_1 is the minimum w within the set W.

Note that we are choosing $\{c[w^t]\}_{t=0}^{\infty}$, where $w^t = (w_0, w_1, \ldots, w_t)$—the history of income realizations up to time t. The sequence problem would give the F.O.C.:

$$\beta^t u'(c[w^t]) = \lambda[w^t] \frac{1}{(1+r)^t},$$

where $\lambda[w^t]$ is the Lagrange multiplier on the budget constraint given a particular path of $w(t)$ is realized up to t.

Interpretation: the (discounted) marginal utility of consumption after history w^t is equated to the (discounted) marginal utility of money after history w^t, $\lambda[w^t]$.

Recursive formulation

Note that the flow budget constraint is

$$a(t+1) = (1+r)\left[a(t) + w(t) - c(t)\right].$$

We can write the problem in recursive formulation as:

$$V(a,w) = \max_{a' \in [-b_1,(1+r)(a+w)]} \left[u\left(a + w - \frac{1}{1+r} a' \right) + \beta E V(a', w') \right],$$

where we've used the fact that $w(t)$ is distributed independently over time, so that the continuation value is not conditioned on the current w.

The F.O.C. with respect to a':

$$u'(c(t)) = (1+r)\beta E_t \frac{\partial V(a(t+1), w(t+1))}{\partial a}.$$

The Envelope condition:

$$\frac{\partial V(a(t), w(t))}{\partial a(t)} = u'(c(t)).$$

Iterating the Envelope condition forward and plugging into the F.O.C. (taking expectations where appropriate), we obtain the Euler equation:

$$u'(c(t)) = \beta(1+r)E_t u'(c(t+1)).$$

Let the utility function be quadratic (Hall 1978): $u(c) = \phi c - \frac{1}{2}c^2$. Note that $u'(c) = \phi - c$: we require that ϕ is sufficiently large in the range of admissible c. The Euler equation becomes:

$$\begin{aligned} \phi - c(t) &= \beta(1+r)E_t[\phi - c_{t+1}] \\ c(t) &= \phi\,(1 - \beta(1+r)) + \beta(1+r)E_t c(t+1). \end{aligned}$$

If $\beta(1+r) = 1$, $E_t \Delta c(t+1) = 0$, where $\Delta \equiv 1 - L$.

Expected consumption should be the same as today's consumption, that is, consumption is a martingale.

Statistical digression

A random variable $z(t)$ is a martingale with respect to some information set Ω_t if $E\left[z(t+1)|\Omega_t\right] = z(t)$.

A random variable $z(t)$ is a submartingale with respect to some information set Ω_t if $E\left[z(t+1)|\Omega_t\right] \geq z(t)$.

A random variable $z(t)$ is a supermartingale with respect to some information set Ω_t if $E\left[z(t+1)|\Omega_t\right] \leq z(t)$.

Plan

We will next talk about consumption problem when endowments are non-stochastic and consumer wants to smooth consumption over time by the means of self-insurance (purchasing or selling a bond) [LS, sections 16.3 and 16.4].

We will then talk about the basics of time series analysis [handouts]. This will be essential to understand:

The Permanent Income Hypothesis (PIH)—self-insurance with uncertain income [handouts].

Then we will talk about consumption problem when the complete array of state-contingent assets is available and income varies unpredictably over time and "states of nature" [LS, ch. 8].

Then we will switch to consumption-based models of asset pricing [LS, ch. 13].