CHAPTER 8: EQUILIBRIUM WITH COMPLETE MARKETS

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Equilibrium in pure exchange, infinite horizon economies, with agents facing stochastic endowments.

Two market structures:

1. Arrow-Debreu structure—complete markets in dated contingent claims (time τ, state vector s^{τ}), all traded at time 0.

2. Sequential, complete, market structure: Arrow securities traded one period in advance; trades occur at each $t \geq 0$.

These two market structures result in identical consumption allocations.

MARKET SETTING

At $t \geq 0$, a stochastic event s_t is drawn from the state space S.

Denote the history of events up to and inclusive the one at t as $s^t = (s_0, s_1, \dots, s_t)$, with the *unconditional* probability of observing s^t, $\pi_t(s^t)$.

The *conditional* probability of observing s^t, given s^τ was realized, is $\pi_t(s^t | s^\tau)$, $t \geq \tau$.

s^t is publicly observable.

Assume $\pi_0(s_0) = 1$, s_0 given.

MARKET SETTING—CONTD.

- Agents: $i = 1, 2, \ldots, I$.
- (Stochastic) Endowments of agent i at t: $y_t^i(s^t)$.
- At time 0, household i purchases a history dependent consumption plan, $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$. Ranks consumption streams with:

$$U^i = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i(s^t)\right] \pi_t(s^t).$$

- u—C^2, strictly concave, $c \geq 0$.
- A feasible allocation must satisfy:

$$\sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \quad \text{all} \quad t, s^t.$$

EQUILIBRIUM OUTCOMES AND HISTORY DEPENDENCE

Individual i's endowment history up to time t can be described as: $y_0^i(s_0), y_1^i(s^1), y_2^i(s^2), \dots, y_t^i(s^t)$. (Note the general dependence of the time t endowment realization on the entire history of stochastic process, preceding time t.)

- In the complete markets models, consumption allocation *does not* depend on individual history, and is *only* a function of the aggregate endowment.

PARETO PROBLEM

Solve the planner's problem:

$$W = \max \sum_{i=1}^{I} \lambda_i U(c^i) \tag{8.4.1}$$

$$\text{s.t.} \quad \sum_i c_t^i(s^t) \leq \sum_i y_t^i(s^t), \tag{8.2.2}$$

where the Pareto weights $\lambda_i \geq 0$, $i = 1, \ldots, I$.

PARETO PROBLEM—CONTD.

Assign non-negative Lagrange multipliers $\theta_t(s^t)$ for each t and s^t constraint (8.2.2), and form the planner's Lagrangian:

$$L = \sum_{t=0} \sum_{s^t} \left[\sum_i \lambda_i \beta^t \pi_t(s^t) u\left[c_t^i(s^t)\right] + \theta_t(s^t) \sum_i \left[y_t^i(s^t) - c_t^i(s^t)\right] \right]$$

For any i, the FOC with respect to $c_t^i(s^t)$ is:

$$\beta^t u'\left[c_t^i(s^t)\right] \pi_t(s^t) = \lambda_i^{-1} \theta_t(s^t)$$

EFFICIENT ALLOCATION

Thus, the ratio of the FOCs for i and 1 is:

$$\frac{u'\left(c_t^i(s^t)\right)}{u'(c_t^1(s^t))} = \frac{\lambda_1}{\lambda_i},$$

$u'(c_t^i(s^t)) = \lambda_i^{-1}\lambda_1 u'(c_t^1(s^t))$

$$c_t^i(s^t) = (u')^{-1}\left[\lambda_i^{-1}\lambda_1 u'(c_t^1(s^t))\right] \tag{8.4.3}$$

$$\sum_i c_t^i(s^t) = \sum_i (u')^{-1}[\lambda_i^{-1}\lambda_1 u'(c_t^1(s^t))] = \sum_i y_t^i(s^t).$$

Thus, $c_t^1(s^t)$ is a function of the aggregate endowment *only*, and *not* dependent on the specific history s^t leading to that outcome, *nor* on the realization of individual endowments.

That is,

$$c_t^i(s^t) = c_\tau^i(s^\tau) \quad \text{when} \quad \sum_i y_t^i(s^t) = \sum_i y_\tau^i(s^\tau).$$

TIME 0 TRADING: ARROW-DEBREU SECURITIES

- Arrow-Debreu securities: claims to consumption for each possible date t, history s^t.
- What is the optimal allocation in a competitive economy with Arrow-Debreu securities?
- Assume: a complete set of state-contingent securities traded at time 0.
- The price of a security at time 0 (now) for 1 unit of consumption to be delivered at t if s^t happens is $q_t^0(s^t)$.

HOUSEHOLD PROBLEM

Household chooses $c^i(s^t)$ to

$$\max U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u\left[c_t^i(s^t)\right] \pi_t(s^t) \tag{8.2.1}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t). \tag{8.5.1}$$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^i(s^t).$$

- No expectations: the household is making all of its lifetime trades at time 0 at a well-defined price vector for all Arrow-Debreu commodities.
- Household buys claims to different "contingent" consumption bundles: delivered if s^t is realized.

Lagrangian for household i is:

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u \left[c_t^i(s^t) \right] \pi_t(s^t) +$$

$$\mu_i \left[\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \right].$$

F.O.C.:

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \mu_i q_t^0(s^t),$$

$$\text{or } \beta^t u' \left[c_t^i(s_t) \right] \pi_t(s^t) = \mu_i q_t^0(s^t) \qquad (8.5.4)$$

COMPETITIVE EQUILIBRIUM

DEFINITIONS. A price system is a sequence of functions $\{q_t^0(s^t)\}_{t=0}^{\infty}$. An allocation is a list of sequences of functions $c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$, one sequence for each i.

DEFINITION. A competitive equilibrium is a *feasible* allocation such that, given the price system, the allocation solves each household's problem.

EQUILIBRIUM ALLOCATIONS

Equation (8.5.4) implies for any i and j: $\frac{u'\left[c_t^i(s^t)\right]}{u'\left[c_t^j(s^t)\right]} = \frac{\mu_i}{\mu_j}$(8.5.5).

That is, the ratios of marginal utilities for any consumers i and j are constant across all dates and state histories.

An equilibrium allocation solves (8.2.2), (8.5.1), and (8.5.5).

(8.5.5) implies for any two consumers i and 1:
$c_t^i(s^t) = (u')^{-1}\left[u'\left(c_t^1(s^t)\right)\frac{\mu_i}{\mu_1}\right]$.

Summing this expression over i and utilizing (8.2.2) at equality, we obtain:

$$\sum_i (u')^{-1}\left[u'\left[c_t^1(s^t)\right]\frac{\mu_i}{\mu_1}\right] = \sum_i c_t^i(s^t) = \sum_i y_t^i(s^t).$$

COMPETITIVE EQUILIBRIUM

The competitive equilibrium allocation is a function of the realized aggregate endowment and depends neither on the specific history leading to that outcome nor on the realizations of individual endowments: $c_t^i(s^t) = c_\tau^i(s^\tau)$ for all s^t and s^τ for which $\sum_i y_t^i(s^t) = \sum_i y_\tau^i(s^\tau)$.

Units of the price system are arbitrary. Let $q_0^0(s^0) = 1$; the rest of the prices will be quoted in the units of time 0 goods.

A competitive equilibrium allocation is a *particular* Pareto optimal allocation, the one for which:

- $\lambda_i = \mu_i^{-1}$.
- The shadow prices $\theta_t(s^t) = q_t^0(s^t)$.

SEQUENTIAL TRADING OF ARROW SECURITIES

It can be shown that one-period Arrow securities are enough to make the markets complete if the markets are reopened each day, and the trades on *one-period ahead state-contingent* consumption claims occur. This sequential trading attains the same allocation as the competitive equilibrium in the economy with time zero trading of state-contingent claims.

We can price different (redundant) assets using the prices for contingent claims from our competitive complete markets equilibrium.

EXAMPLE 8.6.1. RISK SHARING WITH CRRA PREFERENCES

Utility function: $U\left[c_t^i(s^t)\right] = \frac{c_t^i(s^t)^{1-\gamma}}{1-\gamma}$, $\gamma > 0$.

Maximizing the expected sum of discounted utilities over all t, s^t s.t. the individual budget constraint, gives the following equilibrium condition for agents i and 1 (an application of (8.5.5)):

$$c_t^i(s^t) = c_t^1(s^t)\left(\frac{\mu_i}{\mu_1}\right)^{-\frac{1}{\gamma}}.$$

Summing over i,

$$\sum_i c_t^i(s^t) = (\mu_1)^{\frac{1}{\gamma}} c_t^1(s^t) \sum_i (\mu_i)^{-\frac{1}{\gamma}} = \sum_i y_t^i(s^t),$$

and so

$$c_t^1(s^t) = (\mu_1)^{-\frac{1}{\gamma}}\left[\sum_i (\mu_i)^{-\frac{1}{\gamma}}\right]^{-1} \sum_i y_t^i(s^t) = const \sum_i y_t^i(s^t).$$

Thus, individual consumption at t, s^t is *perfectly* correlated with the *aggregate endowment* and *aggregate consumption*.

RISK SHARING. SUMMARY

- Consumption of agents i and j are constant fractions of one another, independent of t, s^t.
- Conditional on time t, history s^t, $c_t^i(s^t)$ is independent of the individual endowment at t, s^t, for all i.
- There is an extensive cross-history, cross-time consumption smoothing. Constant-fraction consumption allocation comes from: 1) the complete markets assumption; 2) a homothetic utility function.

EXAMPLE (8.6.2): NO AGGREGATE UNCERTAINTY

Assume $s_t \in [0,1]$; s^t is a stochastic event; $i = 1, 2$; $y_t^1(s^t) = s_t$; $y_t^2(s^t) = 1 - s_t$.

Aggregate endowment at t, s^t:

$$\sum_i y_t^i(s^t) = s_t + (1 - s_t) = 1.$$

Thus, it immediately follows from (8.5.7) that $c_t^1(s^t)$ is constant for each t, s^t. $c_t^2(s^t) = 1 - c_t^1(s^t)$ is also constant $\forall t, s^t$.

Equilibrium allocation:

$$c_t^i(s^t) = \overline{c}^i.$$

From (8.5.4) it follows

$$\beta^t \pi_t(s^t) \frac{u'\left(\bar{c}^i\right)}{\mu_i} = q_t^0(s^t), \quad \forall i, t, s^t \tag{8.6.2}$$

Agent i's b.c.:

$$\sum_t \sum_{s^t} q_t^0(s^t)[\bar{c}^i - y_t^i(s^t)] = 0,$$

or

$$\bar{c}^i \sum_t \sum_{s^t} q_t^0(s^t) - \sum_t \sum_{s^t} q_t^0(s^t) y_t^i(s^t) = 0.$$

Plug the definition for $q_t^0(s^t)$ from (8.5.4), to obtain:

$$\overline{c}^i \frac{u'(\overline{c}^i)}{\mu_i} \sum_t \sum_{s^t} \pi_t(s^t)\beta^t - \frac{u'(\overline{c}^i)}{\mu_i} \sum_t \sum_{s^t} \pi_t(s^t)\beta^t y_t^i(s^t) = 0$$

$$\overline{c}^i \sum_t \beta^t \sum_{s^t} \pi_t(s^t) - \sum_t \sum_{s^t} \pi_t(s^t)\beta^t y_t^i(s^t) = 0$$

$$c^i = (1-\beta) \sum_t \sum_{s^t} \pi_t(s^t)\beta^t y_t^i(s^t) \tag{8.6.3}$$

Is this allocation feasible? For each t, s^t:

$$\begin{aligned}
\sum_i \overline{c}^i &= (1-\beta) \sum_i \sum_t \sum_{s^t} \pi_t(s^t)\beta^t y_t^i(s^t) \\
&= (1-\beta) \sum_t \sum_{s^t} \sum_i \pi_t(s^t)\beta^t y_t^i(s^t) \\
&= (1-\beta) \sum_t \sum_{s^t} \pi_t(s^t)\beta^t * 1 \\
&= (1-\beta) \sum_t \beta^t \sum_{s^t} \pi_t(s^t) \\
&= (1-\beta)/(1-\beta) = 1.
\end{aligned}$$

In this example, consumer perfectly smoothes consumption over time and across histories since aggregate endowment does not fluctuate across time.

EXAMPLE: PERIODIC ENDOWMENT PROCESSES

$s_t \in [0,1]$, s_t deterministic. Economy starts at $t = 0$. Agent 1's endowment sequence: $1, 0, 1, 0, 1, \ldots$. Agent 2's endowment sequence: $0, 1, 0, 1, 0, \ldots$. $\pi_t(s^t) = 1$ since s^t is deterministic.

(8.5.4) implies:

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \frac{u'(\bar{c}^i)}{\mu_i} = \beta^t \frac{u'(\bar{c}^i)}{\mu_i}.$$

Normalize $q_0^0(s_0) = 1$, i.e., set the time-0 good as a numeraire. Thus, $1 = \beta^0 \frac{u'(\bar{c}^i)}{\mu_i} = \frac{u'(\bar{c}^i)}{\mu_i}$. It follows that

$$q_t(s^t) = \beta^t.$$

(8.6.3) implies:

$$\begin{aligned}\bar{c}^i &= (1-\beta)\sum_t \sum_{s^t} \pi_t(s^t)\beta^t y_t^i(s^t) \\ &= (1-\beta)\sum_t \sum_{s^t} \beta^t y_t^i(s^t).\end{aligned}$$

$\bar{c}^1 = (1-\beta)(1+\beta*0+1*\beta^2+0*\beta^3+1*\beta^4+\dots) = (1-\beta)\frac{1}{1-\beta^2} = \frac{1}{1+\beta}$.

$\bar{c}^2 = 1-\bar{c}^1 = \frac{\beta}{1+\beta}$.

Note that a lifetime richer agent 1 consumes more.

Asset pricing in complete markets

In complete markets, an asset will offer a bundle of history-contingent claims, each component of which has been already priced in the market. In this case, the asset is viewed as *redundant*.

Pricing redundant assets

An asset brings $\{d_t(s^t)\}_{t=0}^{\infty}$—a stream of claims on t, s^t consumption (in physical units). (Think of dividends here.)

The time 0 price of this asset in terms of time 0, history s_0 consumption must be:

$$p_0^0(s_0) = \sum_{t=0}^{\infty} \sum_{s^t} d_t(s^t) q_t^0(s^t).$$

Why? We can replicate this asset by purchasing consumption bundles $\{d_t(s^t)\}_{t=0}^{\infty}$ at the total cost $\sum_{t=0}^{\infty} \sum_{s^t} d_t(s^t) q_t^0(s^t)$. Hence, the asset must cost $p_0^0(s_0)$.

A *riskless consol* brings 1 unit of consumption in *each* t, regardless of s^t.

To replicate its payoffs, we need to purchase $q_t^0(s^t)$ units of claims for each t, s^t. The price of this asset, therefore, is:

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t).$$

A *riskless strip* pays $d_\tau = 1$ if $\tau = t \geq 0$, 0—otherwise. Thus, the strips pay off only at time τ, regardless of the history realization s^τ.

To replicate the asset's payoffs, we need to purchase, at $t = 0$, $q_\tau^0(s^\tau)$ units of consumption for time τ, all histories s^τ. Thus, the price of the strip must be

$$\sum_{s^\tau} q_\tau^0(s^\tau).$$

Pricing tail assets

Tail assets pay $\{d_t(s^t)\}_{t \geq \tau, \tau \geq 1}^{\infty}$. At time 0, this asset will cost

$$p_\tau^0(s^\tau) = \sum_{s^t | s^\tau} \sum_{t \geq \tau} d_t(s^t) q_t^0(s^t).$$

Units of this price are in terms of time 0, state $s^0 = s_0$ goods.

Quote the price in *units of time τ, state s^τ goods*. Divide the price by $q_\tau^0(s^\tau)$, to obtain:

$$p_\tau^\tau(s^\tau) \equiv \frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t \geq \tau} \sum_{s^t | s^\tau} d_t(s^t) \frac{q_t^0(s^t)}{q_0^\tau(s^\tau)}. \tag{8.7.4}$$

- $p_\tau^0(s^\tau)$ measures the amount of goods at time 0 state s^0 it costs to purchase the asset (# of time 0 goods per 1 asset).
- $q_\tau^0(s^\tau)$ measures the amount of goods at time 0 state s^0 it costs to purchase the claim to one good at time τ state s^τ (# of time 0 goods per 1 unit of goods of time τ state s^τ).
- It follows that $\frac{p_\tau^0(s^\tau)}{q_\tau^0(s^\tau)}$ measures the amount of goods of time τ state s^τ it costs to purchase the asset.

Define $p^\tau_\tau(s^\tau) = \frac{p^0_\tau(s^\tau)}{q^0_\tau(s^\tau)}$—the price of the tail asset in terms of the time τ, history s^τ consumption goods.
Note that in (8.7.4)

$$q^\tau_t(s^t) \equiv \frac{q^0_t(s^t)}{q^\tau_0(s^\tau)} = \frac{\mu_i^{-1}\beta^t u'\left[c^i_t(s^t)\right]\pi_t(s^t)}{\mu_i^{-1}\beta^\tau u'[c^i_\tau(s^\tau)]\pi_\tau(s^\tau)}$$
$$= \beta^{t-\tau}\frac{u'[c^i_t(s^t)]}{u'[c^i_\tau(s^\tau)]}\frac{\pi_t(s^t)}{\pi_\tau(s^\tau)}. \qquad (8.7.5)$$

Since the LHS of the equation is *not* individual-dependent, the RHS should not be too. Note that $\frac{\pi_t(s^t)}{\pi_\tau(s^\tau)} = \frac{\pi(s^t|s^\tau)\pi(s^\tau)}{\pi(s^\tau)}$.
Thus,

$$p^\tau_\tau(s^\tau) = \sum_{t\geq\tau}\sum_{s^t|s^\tau}\left[\beta^{t-\tau}\frac{u'[c^i_t(s^t)]}{u'[c^i_\tau(s^\tau)]}\pi_t(s^t|s^\tau)\right]d_t(s^t). \qquad (8.7.6)$$

Pricing one-period returns

Replace t with $\tau + 1$ in (8.7.5), to obtain:

$$q^{\tau}_{\tau+1}(s^{\tau}) = \beta \frac{u'[c^{i}_{\tau+1}(s^{\tau+1})]}{u'[c^{i}_{\tau}(s^{\tau})]} \pi_{\tau+1}(s^{\tau+1}|s^{\tau}).$$

If we want to find the price at time τ, history s^{τ} of a claim to a random payoff $\omega(s_{\tau+1})$, we'd use the following formula:

$$p^{\tau}_{\tau}(s^{\tau}) = \sum_{s_{\tau+1}} q^{\tau}_{\tau+1}(s^{\tau+1})\omega(s_{\tau+1}) = E_{\tau}\left[\beta \frac{u'[c_{\tau+1}]}{u'[c_{\tau}]}\omega(s_{\tau+1})\right]. \tag{8.7.7}$$

Let the one-period gross return on the asset with payoff $\omega(s_{\tau+1})$ be $R_{\tau+1} \equiv \frac{\omega(s_{\tau+1})}{p_\tau^\tau(s^\tau)}$.

Then (8.7.7) becomes:

$$E_\tau\left[\beta\frac{u'\left[c_{\tau+1}\right]}{u'\left[c_\tau\right]}R_{\tau+1}\right] = E_\tau\left[m_{\tau+1}R_{\tau+1}\right] = 1,$$

where $m_{\tau+1} = \beta\frac{u'[c_{\tau+1}]}{u'[c_\tau]}$ is called a stochastic discount factor.

Sequential trading: Arrow securities

- At each date $t \geq 0$, trades occur in a set of claims to one-period-ahead state-contingent consumption.

- With a full array of one-period-ahead claims, the sequential trading arrangement attains the same competitive allocation as we described earlier.

- If trading takes place in period 0, what is the implied wealth of household i at time t after history s^t? It is the the value of household's purchased claims net of its outstanding liabilities.
- When history s^t is realized discard all claims and liabilities conditional on another initial history.
- Household's net claim of goods at τ is $c_\tau(s^\tau) - y_\tau(s^\tau)$.
- Household's wealth in terms of the date t, history s^t consumption good is

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) \left[c_\tau^i(s^\tau) - y_\tau^i(s^\tau) \right] . \tag{8.8.1}$$

Notice that

$$\sum_{i=1}^{I} \Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) \underbrace{\sum_{i=1}^{I} \left[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)\right]}_{=0 \text{ by feasibility}}, \text{for each } \; t, s^t.$$

We can match up the time t, history s^t wealth of the household in the sequential economy with the tail wealth $\Upsilon_t^i(s^t)$ from the Arrow-Debreu economy as in equation (8.8.1).

Debt limits

"Natural debt limits:" it should be feasible for the consumer to repay his state-contingent debt in every possible state.

Let $q_\tau^t(s^\tau)$ be the price of the good at time τ, history s^τ in units of the good at time t, history s^t. The value of the tail of the endowment sequence at time t is

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau). \qquad (8.8.2)$$

$A_t^i(s^t)$ is the natural debt limit at time t and history s^t.

At each date $t \geq 0$, households trade claims to date t consumption, whose payment is contingent on the realization of s_{t+1}.

- $\tilde{a}_t^i(s^t)$—the claims to time t consumption that household brings to time t, history s^t.
- $\tilde{Q}_t(s_{t+1}|s^t)$—the price of one unit of time $t+1$ consumption contingent on realization of s_{t+1} at $t+1$, given the history at t is s^t.
- The time t history s^t budget constraint is

$$\underbrace{\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}, s^t)}_{\text{time } t \text{ history } s^t \text{ expenditures}} \leq \underbrace{y_t^i(s^t) + \tilde{a}_t^i(s^t)}_{\text{time } t \text{ history } s^t \text{ resources}} \tag{8.8.3}$$

State-by-state borrowing constraints

$$\tilde{a}^i_{t+1}(s^{t+1}) \geq -A^i_{t+1}(s^{t+1}). \tag{8.8.4}$$

The Lagrangian is

$$\begin{aligned} L^i = \sum_{t=0}^{\infty} \sum_{s^t} \{ & \beta^t u\left(\tilde{c}^i_t(s^t)\right) \pi_t(s^t) \\ & + \eta^i_t(s^t) \left[y^i_t(s^t) + \tilde{a}^i_t(s^t) - \tilde{c}^i_t(s^t) - \sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t) \tilde{Q}_t \left(s_{t+1} | s^t\right) \right] \\ & + \sum_{s_{t+1}} \nu^i_t(s^t; s_{t+1}) \left[A^i_{t+1}(s_{t+1}, s^t) + \tilde{a}^i_{t+1}(s_{t+1}, s^t) \right] \}, \end{aligned}$$

for a given $\tilde{a}^i_0(s_0)$.

FOCs w.r.t. $\tilde{c}_t^i(s^t)$ and $\{\tilde{a}_{t+1}^i(s_{t+1}, s^t)\}_{s_{t+1}}$ are

$$\beta^t u' \left[\tilde{c}_t^i(s^t)\right] \pi_t(s^t) - \eta_t^i(s^t) = 0 \quad (8.8.5a)$$

$$-\eta_t^i(s^t) \tilde{Q}_t(s_{t+1}|s^t) + \nu_t^i(s^t; s_{t+1}) + \eta_{t+1}^i(s_{t+1}, s^t) = 0, \quad (8.8.5b)$$

for all t, s^t, and s_{t+1}. At the optimum, natural borrowing constraints are not binding, and so $\nu_t^i(s^t; s_{t+1}) = 0$. Thus,

$$\tilde{Q}_t(s_{t+1}|s^t) = \beta \frac{u' \left[\tilde{c}_{t+1}^i(s^{t+1})\right]}{u' \left[\tilde{c}_t^i(s^t)\right]} \pi_t(s^{t+1}|s^t), \quad (8.8.6)$$

for all t, s^t, and s_{t+1}.

Equivalence of allocations

AD equilibrium is also a sequential-trading equilibrium with a particular initial distribution of wealth. Make a guess that

$$\begin{aligned} q^0_{t+1}(s^{t+1}) &= \tilde{Q}_t(s_{t+1}|s^t)q^0_t(s^t) \\ \tilde{Q}_t(s_{t+1}|s^t) &= \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = q^t_{t+1}(s^{t+1}). \end{aligned} \tag{8.8.7}$$

For the Arrow-Debreu economy, the following Euler equation should hold

$$\frac{\beta u'\left[c^i_{t+1}(s^{t+1})\right]\pi\left(s^{t+1}|s^t\right)}{u'\left[c^i_t(s^t)\right]} = \frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = \tilde{Q}_t(s_{t+1}|s^t) \tag{8.8.8}$$

The equation is equivalent with the sequential-trading FOC (8.8.6). We need to establish that $\tilde{c}^i_t(s^t) = c^i_t(s^t)$.

Choose the initial wealth in the sequential-trading equilibrium so that this equilibrium duplicates the Arrow-Debreu equilibrium. Let $\tilde{a}_0^i(s_0)$ for all i—households rely on their endowment stream to finance its consumption.

Need to show that household i will be able to finance $\{c_t^i(s^t)\}$ and there will be no room to improve on that allocation.

Guess that

$$\tilde{a}^i_{t+1}(s_{t+1}, s^t) = \Upsilon^i_{t+1}(s^{t+1}), \text{for all } \; s_{t+1}.$$

The value of this portfolio is

$$\begin{aligned}
\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) &= \sum_{s^{t+1}|s^t} \Upsilon^i_{t+1}(s^{t+1}) q^t_{t+1}(s^{t+1}) \\
&= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} \sum_{s^{t+1}|s^t} q^t_{t+1}(s^{t+1}) q^{t+1}_\tau(s^\tau) \left[c^i_\tau(s^\tau) - y^i_\tau(s^\tau) \right] \\
&= \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} q^t_\tau(s^\tau) \left[c^i_\tau(s^\tau) - y^i_\tau(s^\tau) \right]. \qquad (8.8.9)
\end{aligned}$$

At time 0, the budget constraint (8.8.3) implies

$$\tilde{c}_0(s_0) + \sum_{t=1}^{\infty}\sum_{s^t} q_t^0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] = y_0^i(s_0) + \underbrace{\tilde{a}_0^i(s_0)}_{=0}$$

$$\tilde{c}_0(s_0) + \sum_{t=1}^{\infty}\sum_{s^t} q_t^0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right] = y_0^i(s_0) - c_0^i(s_0) + c_0^i(s_0)$$

$$\underbrace{\tilde{c}_0(s_0) + \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\left[c_t^i(s^t) - y_t^i(s^t)\right]}_{=0 \text{ since } u'>0} = c_0^i(s_0).$$

Thus, the proposed portfolio is affordable and in period 0 and the associated consumption level is the same as in the Arrow-Debreu competitive equilibrium.

From (8.8.9),

$$\begin{aligned}\sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t)\tilde{Q}_t(s_{t+1}|s^t) &= \sum_{\tau=t}^{\infty}\sum_{s^t} q^t_\tau(s^\tau)\left[c^i_\tau(s^\tau) - y^i_\tau(s^\tau)\right] \\ &\quad - \underbrace{q^t_t(s^t)}_{=1}\left[c^i_t(s^t) - y^i_t(s^t)\right] \\ &= \Upsilon^i_t(s^t) - \left[c^i_t(s^t) - y^i_t(s^t)\right].\end{aligned}$$

From (8.8.3) it follows,

$$\begin{aligned}\tilde{c}^i_t(s^t) + \Upsilon^i_t(s^t) - \left[c^i_t(s^t) - y^i_t(s^t)\right] &= y^i_t(s^t) + \tilde{a}^i_t(s^t) \\ \tilde{c}^i_t(s^t) + \Upsilon^i_t(s^t) &= c^i_t(s^t) + \tilde{a}^i_t(s^t).\end{aligned}$$

Thus, when $\tilde{a}^i_t(s^t) = \Upsilon^i_t(s^t)$, $\tilde{c}^i_t(s^t) = c^i_t(s^t)$, for all t, s^t.