Chapter 13: Asset Pricing

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Agenda

We want to:

- Price assets *without* specifying a full set of contingent claims/securities.

- Derive restrictions on asset prices, asset returns, and consumption allocations from an agent's optimization problem.

- Study the equity premium puzzle, and Lucas's asset pricing model.

Asset Euler Equations. Set-Up

Assume 2 assets—bond and equity holdings. Can use these assets to transfer wealth over time.

Household's optimization provides restrictions on asset prices and consumption, that should *be satisfied* when additional assets are introduced.

A household has a stock of wealth $A_t > 0$ at t, and wants to:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}), \;\; 0 < \beta < 1.$$

E_t is the mathematical expectation conditional on information at t; $u(\cdot) \in C^2$, $u' > 0, u'' < 0$.

Set-Up. Contd.

One-period bonds earn a risk-free real gross interest R_t, measured in units of consumption goods at $t+1$ per 1 unit of consumption goods at t.

L_t—the gross payout on the agent's bonds held between t and $t+1$, payable at $t+1$. Present value at t is $R_t^{-1}L_t$. $L_t < 0$, if the consumer issues bonds (borrows funds) at t.

N_t—consumer's holdings of equity shares between t and $t+1$. $N_t < 0$, if sells equity short (sell at t without having equity).

$L_t \geq -b_L$, $N_t \geq -b_N$, $b_L \geq 0$, $b_N \geq 0$.

One share of equity entitles an owner to the stochastic dividend at t, y_t. Let the share price *net* of the dividend be p_t.

Euler Equations

Budget constraint and A_{t+1} are:

$$c_t + R_t^{-1} L_t + p_t N_t \leq A_t \tag{13.2.2}$$

$$A_{t+1} = L_t + (p_{t+1} + y_{t+1}) N_t \tag{13.2.3}$$

DP problem: states—$y_t, y_{t-1}, \ldots, A_t$; controls—$L_t, N_t$.

The Euler equations:

$$u'(c_t) R_t^{-1} = E_t \beta u'(c_{t+1}) \tag{13.2.4}$$

$$u'(c_t) p_t = E_t \beta u'(c_{t+1})(p_{t+1} + y_{t+1}) \tag{13.2.5}$$

Martingale Theories of Consumption and Stock Prices

If $R_t = R$, $\forall t$, then (13.2.4) implies

$$E_t u'(c_{t+1}) = (\beta R)^{-1} u'(c_t).$$

Conditional on $u'(c_t)$, no other variable in the information set at t should predict $u'(c_{t+1})$. E.g., MU of consumption is a martingale if $R\beta = 1$.

Efficient stock markets hypothesis: the price of a stock should follow a martingale.

Use (13.2.5), to show:

$$p_t = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + y_{t+1}) \right]$$

$$p_t = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} \right) E_t \left(p_{t+1} + y_{t+1} \right) + \beta cov_t \left[\frac{u'(c_{t+1})}{u'(c_t)}, p_{t+1} + y_{t+1} \right]$$

If consumer is risk neutral, $u'(c_t)$ is independent of c_t, and $cov_t[const, p_{t+1} + y_{t+1}] = 0$. Thus, $p_t = \beta E_t \left[p_{t+1} + y_{t+1} \right]$: the price of a share, adjusted for dividends and time discounting, follows a first-order Markov process.

Lucas's Asset Pricing Model

Assumptions/Economy's Structure:

- A large number of identical agents
- Durable goods—identical "trees," one per each person.
- In the beginning of t, each tree yields fruit/dividends y_t.
- Fruit/dividend is non-storable.
- Economy starts at $t = 0$.

- y_t is driven by a Markov process. Aggregate state, $s_t = y_t$, where $Prob(s_{t+1} \leq s' | s_t = s) = F(s', s)$.

- 2 assets in the economy: tree (one share per tree) and a bond. 2 Euler equations at each t.

- Markets clear each t: Bond holdings sum to 0 in the aggregate; equity positions sum to the number of shares/trees in the economy.

- Since agents are identical, can work with a representative agent. Set $c_t = y_t$, $\forall t$, also equal to per capita quantities. This allocation satisfies, e.g., the social planner's choice of consumption allocation for the representative agent:

$$\max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{j=0}^{\infty} \beta^j u(c_t)$$

$$\text{s.t. } c_t \leq y_t.$$

Stock Prices in Lucas's Economy

Plug equilibrium consumption at t and $t+1$ into (13.2.4)–(13.2.5), to obtain:

$$u'(y_t)R_t^{-1} = \beta E_t u'(y_{t+1}) \tag{13.5.1}$$

$$u'(y_t)p_t = E_t\left[\beta u'(y_{t+1})(p_{t+1} + y_{t+1})\right]. \tag{13.5.2}$$

Use (13.5.2), to find:

$$E_t u'(y_{t+1})p_{t+1} = E_t E_{t+1}\beta u'(y_{t+2})y_{t+2} + E_t E_{t+1}\beta u'(y_{t+2})p_{t+2}$$

$$E_t\beta u'(y_{t+1})p_{t+1} = E_t\beta^2 u'(y_{t+2})y_{t+2} + E_t\beta^2 u'(y_{t+2})p_{t+2}$$

$$u'(y_t)p_t = E_t\beta u'(y_{t+1})y_{t+1} + E_t\beta^2 u'(y_{t+2})y_{t+2} + E_t\beta^2 u'(y_{t+2})p_{t+2}$$

$$\vdots$$

$$u'(y_t)p_t = E_t\sum_{j=1}^{\infty}\beta^j u'(y_{t+j})y_{t+j} + E_t\lim_{k\to\infty}\beta^k u'(y_{t+k})p_{t+k} \tag{13.6.1}$$

The Price of a Share

The last (bubble) term on the RHS of (13.6.1) should be zero. Thus, the equilibrium price of the share should be defined from:

$$p_t = E_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(y_{t+j})}{u'(y_t)} y_{t+j} \right] \tag{13.6.2}$$

In words: the share price at t is the expected discounted sum of future dividends with time-varying discount factors.

Since y_t is a Markov process, $p_t = p(s_t)$.

If $u(c_t) = \log(c_t)$, from (13.6.2),
$p_t = E_t \left[\beta \frac{y_t}{y_{t+1}} y_{t+1} + \beta^2 \frac{y_t}{y_{t+2}} y_{t+2} + \beta^3 \frac{y_t}{y_{t+3}} y_{t+3} + \cdots \right] = \frac{\beta}{1-\beta} y_t.$
Work out example 3.

A Closer Look at the CRRA

Utility function: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.
The coefficient of relative risk aversion is

$$\gamma = -\frac{cu''(c)}{u'(c)}.$$

Why?

Consider a gamble:

- to receive $c - \pi$ with certainty, or
- $c + y$ with probability $1/2$, or $c - y$ with probability $1/2$. I.e., $\tilde{y} = y$ with probability $1/2$, and $\tilde{y} = -y$ with probability $1/2$.

What is $\pi(c, y)$ that leaves consumer indifferent between the risky and riskless choices?

$\pi(c, y)$ should solve

$$u(c - \pi) = \frac{1}{2}u(c + y) + \frac{1}{2}u(c - y) = Eu(c + \tilde{y}).$$

CRRA—contd.

Take the first-order Taylor expansion of the LHS about c:
$u(c - \pi) \approx u(c) - \pi u'(c)$.
Take the second-order Taylor expansion of the RHS about c:

$$u(c + \tilde{y}) \approx u(c) + u'(c)\tilde{y} + \frac{1}{2}u''(c)\tilde{y}^2$$

$$Eu(c + \tilde{y}) \approx u(c) + \frac{1}{2}u''(c)E\tilde{y}^2$$

Equate them, to obtain: $\pi(c, y) \approx \frac{1}{2}y^2[-\frac{u''(c)}{u'(c)}]$. For the CRRA utility, $\pi(c, y) \approx \frac{1}{2}y^2\frac{\gamma}{c}$.

$\pi(c, y)$ is the risk premium a CRRA consumer with an initial consumption c will be willing to pay in order to avoid a fair bet of winning or losing y dollars. E.g., for $c = 50,000$, $\gamma = 2$, and $y = 5,000$, $\pi(c, y) = 500$; if $\gamma = 10$, $\pi(c, y) = 2,212$.

The Equity Premium Puzzle

- Mehra and Prescott (1985): aggregate data for 1889–1978.
- The average real yield on risky equity was about 7%, while the average yield on the short term debt was 1%.
- Let the real return on bonds between t and $t+1$ be $1 + r^b_{t+1}$; and the real return on stocks be $1 + r^s_{t+1}$. Lucas's model notation: R_t, and $\frac{y_{t+1}+p_{t+1}}{p_t}$, respectively.
- 2 Euler equations: $1 = \beta E_t \left[(1 + r^i_{t+1}) \frac{u'(c_{t+1})}{u'(c_t)} \right]$, $i = s, b$.

Equity Premium Puzzle–Contd.

Let $\frac{c_{t+1}}{c_t} = \overline{c}\exp[\epsilon_{c,t+1} - \sigma_c^2/2]$

$(1 + r_{t+1}^i) = (1 + \overline{r}^i)\exp[\epsilon_{i,t+1} - \sigma_i^2/2]$.

$\epsilon_{c,t+1}, \epsilon_{b,t+1}, \epsilon_{s,t+1}$ are jointly normally distributed with zero means, and an unrestricted variance-covariance matrix.

Thus, the unconditional version of the Euler equation will read:

$$1 = \beta(1 + \overline{r}^i)\overline{c}^{-\gamma} E\left(\exp[\epsilon_{i,t+1} - \sigma_i^2/2]\exp[-\gamma\epsilon_{c,t+1} + \gamma\sigma_c^2/2]\right)$$

$$1 = \beta(1 + \overline{r}^i)\overline{c}^{-\gamma}\exp(\gamma\sigma_c^2/2)\exp(-\sigma_i^2/2) E\left(\exp[\epsilon_{i,t+1} - \gamma\epsilon_{c,t+1}]\right)$$

$$1 = \beta(1 + \overline{r}^i)\overline{c}^{-\gamma}\exp(\gamma\sigma_c^2/2)\exp(-\sigma_i^2/2)\exp[\sigma_i^2/2 + \gamma^2/2 * \sigma_c^2 - \gamma cov(\epsilon_i, \epsilon_c)]\} \quad (*)$$

$$1 = \beta(1 + \overline{r}^i)\overline{c}^{-\gamma}\exp[\gamma(1 + \gamma)\sigma_c^2 - \gamma cov(\epsilon_c, \epsilon_i)], \qquad (13.12.4)$$

where (*) follows from $E[\exp u_{t+1}] = \exp(\mu_u + 1/2\sigma_u^2)$. For our case, $u_{t+1} = -\gamma\epsilon_{c,t+1} + \epsilon_{i,t+1}$, and so $\sigma_u^2 = \gamma^2\sigma_c^2 + \sigma_i^2 - 2 * cov(\epsilon_c, \epsilon_i)$.

Equity Premium Puzzle

Take logs of (13.12.4), to obtain:

$$\log(1+\bar{r}^i) = -\log\beta + \gamma\log(\bar{c}) - \gamma(1+\gamma)\sigma_c^2/2 + \gamma cov(\epsilon_i, \epsilon_c), \quad i = s, b.$$

The equity premium is:

$$\log(1+\bar{r}^s) - \log(1+\bar{r}^b) = \gamma[cov(\epsilon_c, \epsilon_s) - cov(\epsilon_c, \epsilon_b)].$$

Taking the approximation $\log(1+\bar{r}) \approx \bar{r}$ for small $\bar{r}$, and setting $cov(\epsilon_c, \epsilon_b)$ to zero (as is close to the data):

$$\bar{r}^s - \bar{r}^b \approx \gamma cov(\epsilon_s, \epsilon_c).$$

In the data, $cov(\epsilon_s, \epsilon_c) = 0.00219$. It requires $\gamma = 27$ (!), to have the historical equity premium of 6%. Hence, the equity premium puzzle.

Add-on: Risk Premium for CRRA Utility

Let $\tilde{y} = k\tilde{x}$, where $E\tilde{x} = 0$.

The risk premium, $g(k) = \pi(c, k\tilde{x})$, for taking a small risk is derived from $u(c - g(k)) = Eu(c + k\tilde{x})$.

Totally differentiate both sides with respect to k:
$-g'(k)u'(c - g(k)) = E\tilde{x}u'(c + k\tilde{x})(**)$. Notice that, when $k = 0$, $E\tilde{x}u'(c + k\tilde{x}) = u'(c)E\tilde{x} = 0$, and so $g'(0) = 0$.

Totally differentiate (**) wrt k:
$[g'(k)]^2 u''(c - g(k)) - g''(k)u'(c - g(k)) = E[\tilde{x}]^2 u''(c + k\tilde{x})$.

At $k = 0$, $g(0) = 0$, $g''(0) = -\frac{u''(c)}{u'(c)} E[\tilde{x}]^2$.

Using a Taylor expansion,
$\pi(c, k\tilde{x}) \equiv g(k) \approx g(0) + g'(0)k + \frac{1}{2}g''(0)k^2$.

$\pi(c, k\tilde{x}) \approx -\frac{u''(c)}{u'(c)} E[\tilde{y}]^2$. For our example, $\tilde{x} = \{1, -1\}$ with probability $1/2$, $k = 5,000$.